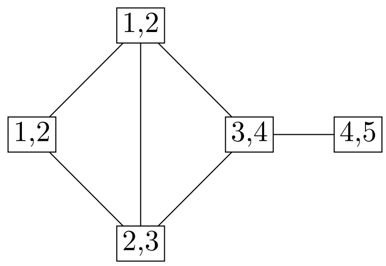
Chapter 5.4 List Coloring

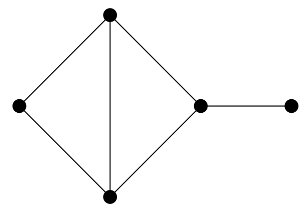
Introduced by Vizing (1976) and independently by Erdős, Rubin, and Taylor (1979).

A *list assignment* of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. The elements of the list $L(v)$ are called *admissible colors* for the vertex v. An *L-coloring* is a mapping $\varphi : V(G) \to \bigcup_v L(v)$ such that

- $\varphi(v) \in L(v)$ for every $v \in V(G)$, and
- $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G.

1: Find a list coloring for the graph below.

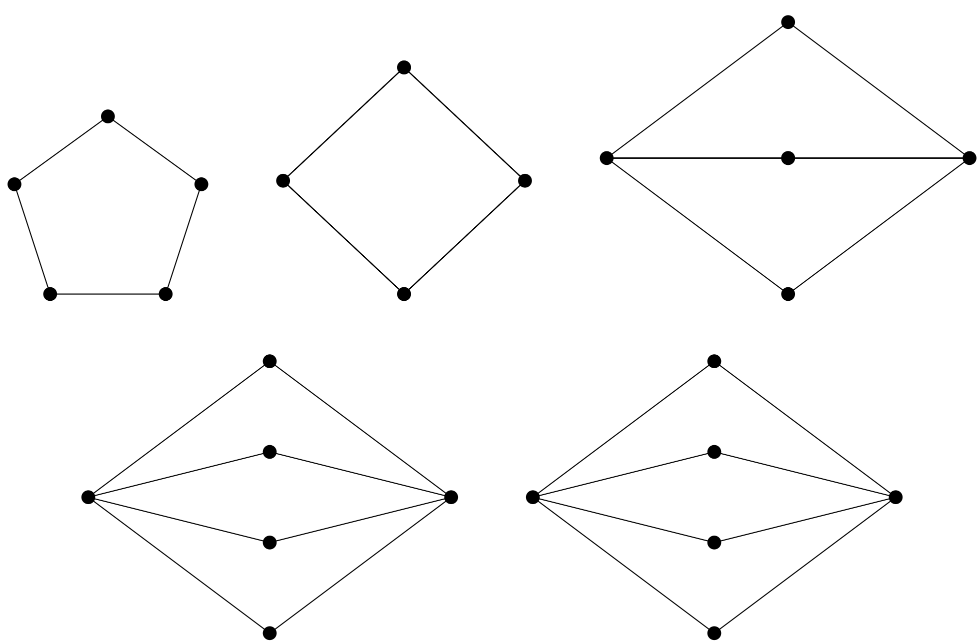




For a graph G we say

- *L-colorable* if G admits an L-coloring
- *k-choosable* if, for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, G is L-colorable.
- *choosability* of a graph G, denoted by $\mathrm{ch}(G)$ or $\chi_\ell(G)$, is the smallest k such that G is k-choosable.

2: What is choosability of the following graphs? (Some depicted twice to allow experiments.)



Erdős, Rubin and Taylor showed that there are bipartite graphs with arbitrary large list chromatic number.

Theorem 1 (Erdős, Rubin and Taylor)**.** *For $m \geq \binom{2k-1}{k}$, the bipartite graph $K_{m,m}$ is not k-choosable.*

3: Prove the theorem. Hint: Use the last graph from previous exercise.

Solution:

Proof. Without loss of generality we can assume $m = \binom{2k-1}{k}$, and we have a bipartition A, B of $K_{m,m}$. Let us use colors from the set $\{1, \ldots, 2k-1\}$, and assign any k-subset of this set to a distinct vertex of A and also to a distinct vertex of B. Now, in a possible list-coloring of $K_{m,m}$ in the bipartition A we must use at least k colors (as otherwise we have vertex in A in which list we miss all used colors on A, and then this vertex cannot be colored). But then in B, we have a vertex with list of these k used colors in A, and we have a conflict in the coloring of this vertex. □

Theorem 2 (Thomassen)**.** *Every planar graph is 5-choosable.*

The next lemma implies the above theorem. Recall that a near-triangulation is a plane graph whose all inner faces are 3-cycles.

Lemma 3. *Let G be a 2-connected near-triangulation and let $C = x_1 x_2 \cdots x_n x_1$ be the outerface. Let L be a list-assignment of G such that $|L(x)| \geq 3$, for $x \in V(C)$, and otherwise $|L(x)| \geq 5$. Suppose that c is an L-coloring of x_1 and x_n. Then, c can be extended to an L-coloring of G.*

4: Prove the lemma by induction.

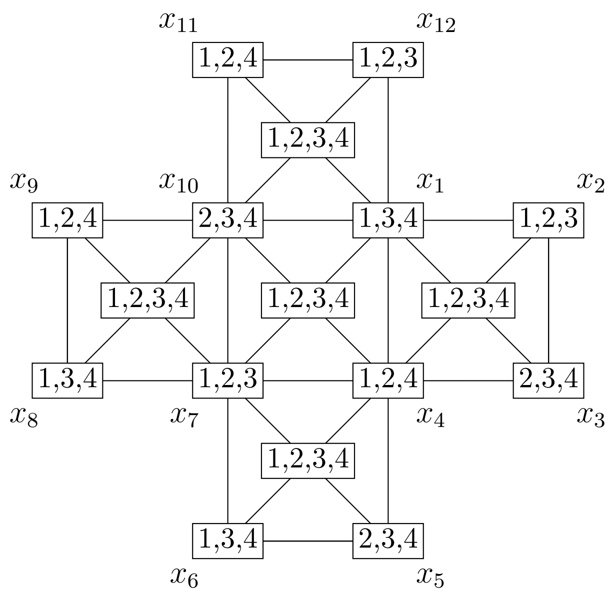
Solution: TODO: It should be split into smaller pieces.

Proof. Suppose that the claim is false and G being a counterexample with $|V(G)| + |E(G)|$ as small as possible. As G is 2-connected, C is a cycle. Observe that G is not a 3-cycle.

First suppose C has a diagonal $x_p x_q$ $(p < q)$, and obviously we can assume $q \neq n$. Let $G_1 = \mathrm{Int}(x_1 \cdots x_p x_q \cdots x_n x_1)$ and $G_2 = \mathrm{Int}(x_p x_{p+1} \cdots x_q x_p)$. By minimality of G, first extend c to G_1, and afterwards the L-coloring of vertices x_p and x_q extend to G_2, and this way we obatin a coloring of G.

Now, we can assume that C is without diagonals. Let $G' = G - x_2$ and let a, b be two distinct colors from $L(x_2) \backslash \{c(x_1)\}$. Let L' be a list-assignment defined as follows: if a vertex $x \in V(G') \backslash \{x_1, x_3\}$ is adjacent with x_2, then let $L'(x) = L(x) \backslash \{a, b\}$, otherwise let $L'(x) = L(x)$. The pair G', L' satisfies the assumptions of the lemma and G' is smaller than G. So we can extend c to a L'-coloring of G'. Let $c(x_2) \in \{a, b\} \backslash \{c(x_3)\}$, and then we obtain that c is a required L-coloring of G. This is a contradiction that establishes the theorem. □

Voigt construct a non-4-choosable planar graph on 238 vertices. Later Mirzakhani (the famous one) such a graph on 63 vertices. A gadget of her construction is depicted below.



5: Show that the graph above is not list-colorable and the graph on the next page is also not list-colorable.

Solution: Observe that this graph is constructed of five blocks, were each block is comprised of a square with an inner vertex connected to all four vertices of the square. Regarding the position of we block we can named as the left, the middle, the right, the upper, and the down square. Observe that a list coloring of vertices of any of these squares requires that at least one pair of diagonal vertices are assigned the same color - otherwise all four of them are colored differently, and in that case there is no available color for the central vertex.

Now for the sake of contradiction assume, we have a list-coloring c. Regarding the color of x_1, we distinguish three cases:

- $c(x_1) = 1$: Then we have $c(x_2) = c(x_4) = 2$, and hence $c(x_5) = c(x_7) = 3$, and from here we derive that $c(x_8) = c(x_{10}) = 4$. Thus, all for vertices of the middle square are colored differently, and there is not free color left for its central vertex.
- $c(x_1) = 3$: Here we argue similarly but we go anti-clockwise. We need to assign $c(x_{10}) = c(x_{12}) = 2$, then $c(x_7) = c(x_9) = 1$, and then $c(x_4) = c(x_6) = 4$. And, again we have no free color for the central vertex of the middle square.
- $c(x_1) = 4$: Then, $c(x_4) = c(x_{10}) = 2$ and so $c(x_7) = 1$ or 3. If $c(x_7) = 1$, then we must assign to x_5, x_6 colors 3, 4, and so we are unable to color the central vertex of the down square. And, in if $c(x_7) = 3$, then similarly we conclude that we must color x_8, x_9 by colors 1, 4, and so we are unable to color the central vertex of the left square.

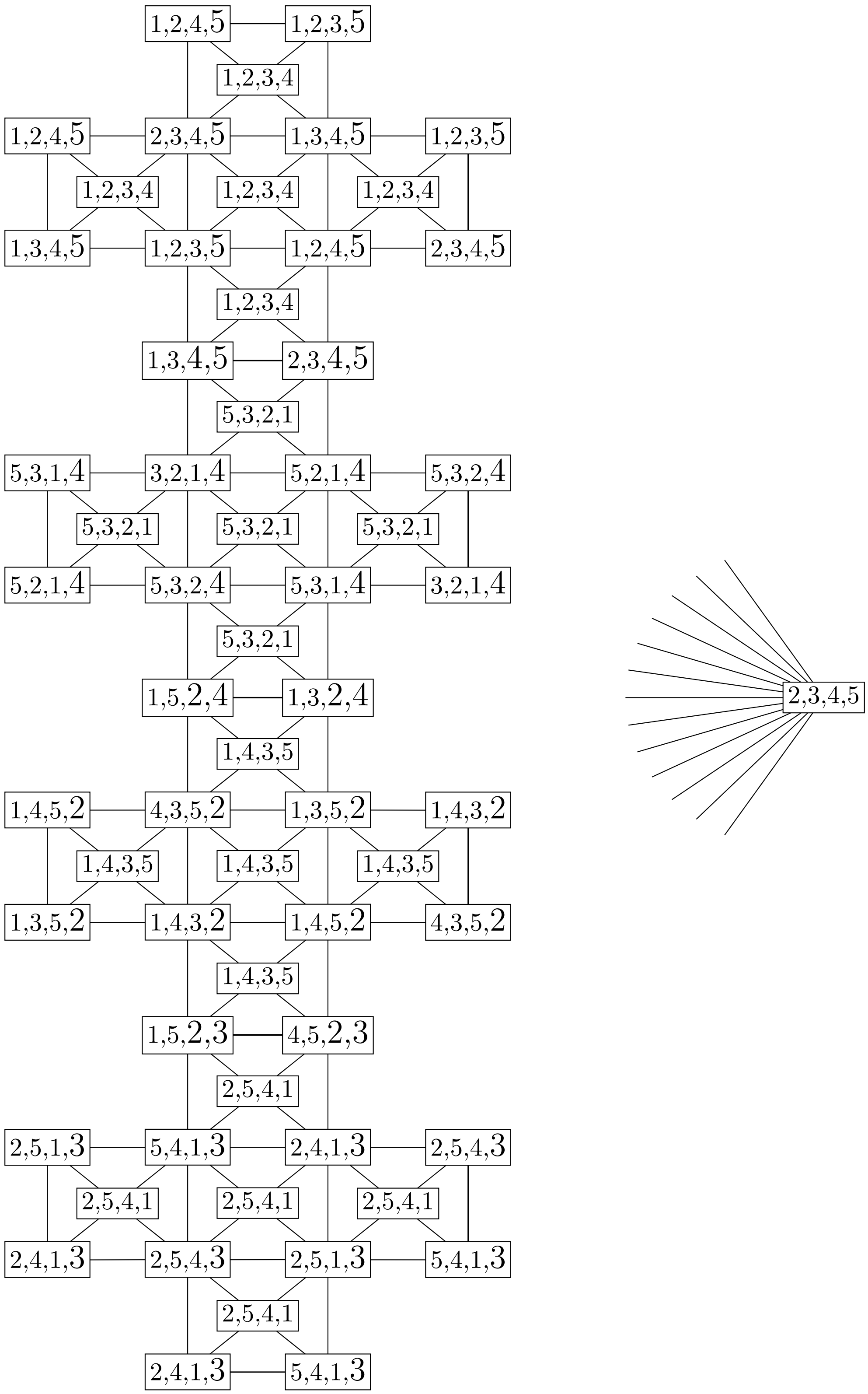


Figure 1: Mirzakhani construction.